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Generalised electrodynamics

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Abstract. The extension of classical electrodynamics proposed by S R Milner is studied in a quaternionic notation. The Lagrangian formulation and the canonical formalism are presented in terms of vector potentials for classical and quantum free field theories. The case of massive vector fields is briefly studied and the possibility of interaction with non-conserved currents is suggested.

1. Introduction

Maxwell electrodynamics in classical as well as in quantum theory exhibits complications not present in other field theories. There is a description in two types of variables, the fields $\boldsymbol{E}$, $\boldsymbol{H}$ or the more convenient potential A^{μ}, which is, however, non-unique and admits gauge transformation. There are difficulties in Lagrangian formulation and, associated with this, singularities in the canonical quantisation procedure. There is also the occurrence of an indefinite inner product in Lorentz covariant description such as the Gupta–Bleuler formalism.

On the other hand one can formulate classical electrodynamics in terms of biquaternions and their functions (Weingarten 1973, Imaeda 1975). Such a formulation is very elegant and presents some unification of physical theories because all the most important fields can be described in terms of the theory of functions of a biquaternion variable.

The quaternionic form of Maxwell electrodynamics suggests some generalisation of the Maxwell equations by adding a new scalar field to common electric and magnetic fields. This was first done by Milner (1963) who suggested that this additional field has non-zero values only 'inside' the elementary particle and that it is responsible for binding forces in classical models of the extended electron, and other elementary particles (Gallop 1975). Such a generalisation of Maxwell theory admits the canonical formulation without any difficulties. One can then perform the procedure of canonical quantisation of the free field theory. In fact we obtain Gupta–Bleuler theory but without a distinguished subspace of 'the physical states'. Finally, we study the case of massive vector fields and present the simple classical model of interaction with a non-conserved current.

2. Electrodynamics in a quaternionic notation

The set of biquaternions Q_{C} is a complex linear algebra generated by four elements

$\{\sigma_\alpha;\ \alpha=0,1,2,3\}$ satisfying the following relations:

$$\sigma_0\equiv 1,\ \sigma_k^2=1$$
$$\sigma_k\sigma_l=\mathrm{i}\sigma_n \tag{2.1}$$

where k, l, n are cyclic permutations of 1, 2, 3. We shall use the following notation and definitions:

$$Q_\mathrm{C}\ni q=q_\alpha\sigma_\alpha=q_0\sigma_o+\boldsymbol{q}\cdot\boldsymbol{\sigma}; \qquad q_\alpha\in\mathrm{C}$$
$$\bar{q}=:q_0\sigma_0-\boldsymbol{q}\cdot\boldsymbol{\sigma}, \qquad q^*=:q_0^*\sigma_0-\boldsymbol{q}^*\cdot\boldsymbol{\sigma},$$
$$q^\dagger=:\overline{q^*}=\bar{q}^*=q_\alpha^*\sigma_\alpha, \qquad \langle q\rangle=:q_0, \tag{2.2}$$

where q_α^* is the complex conjugate of q_α. We have the following important subset of a biquaternion space:

$$Q_\mathrm{R}=\{q\in Q_\mathrm{C};\ q^\dagger=q\}. \tag{2.3}$$

The set Q_R is very convenient for the description of the Minkowski space M_4 (Synge 1972). The isomorphism between Q_R and M_4 is given by

$$M_4\ni x^\mu\mapsto x=x^\mu\sigma_\mu\in Q_\mathrm{R}$$
$$M_4^*\ni x_\mu\mapsto \bar{x}=x^\mu\bar{\sigma}_\mu=x_\mu\sigma_\mu\in Q_\mathrm{R}$$
$$x^\mu y_\mu=\langle x\bar{y}\rangle. \tag{2.4}$$

The proper Lorentz transformation is expressed as

$$x\mapsto x'=axa^\dagger \qquad \bar{x}\mapsto\bar{x}'=a^*\bar{x}\bar{a} \tag{2.5}$$

where $a\bar{a}=a^*a^\dagger=1$, $a\in Q_\mathrm{C}$. The covariant gradient $\partial_0\sigma_0+\nabla\boldsymbol{\sigma}$ is denoted by $\bar{\partial}$ and

$$\bar{\partial}'=a^*\bar{\partial}\bar{a}. \tag{2.6}$$

One can show (Weingarten 1973, Imaeda 1976) that the following equation is equivalent to the Maxwell equations:

$$\partial Z=J \tag{2.7}$$

where $Z=(\boldsymbol{E}+\mathrm{i}\boldsymbol{H})\cdot\boldsymbol{\sigma}$, $\boldsymbol{E}$ is an electric field, $\boldsymbol{H}$ is a magnetic field and J^μ is an external current density:

$$J=J^\mu\sigma_\mu.$$

Milner proposed to modify Maxwell theory by adding the scalar part to the field quaternion Z:

$$Z=Z_0\sigma_0+(\boldsymbol{E}+\mathrm{i}\boldsymbol{H})\cdot\boldsymbol{\sigma}$$
$$Z_0=e+\mathrm{i}h, \qquad e,h\in\mathrm{R}. \tag{2.8}$$

The Maxwell–Milner equations now have the following forms in a common notation

(Milner 1963):

$$\begin{aligned}
&\operatorname{div} \boldsymbol{E} = -\frac{\partial e}{\partial t} + \rho_e \qquad t \equiv x_0,\ c \equiv 1 \\
&\operatorname{div} \boldsymbol{H} = -\frac{\partial h}{\partial t} + \rho_m \\
&\frac{\partial \boldsymbol{H}}{\partial t} + \operatorname{rot} \boldsymbol{E} = -\operatorname{grad} h + \boldsymbol{j}_m \\
&\frac{\partial \boldsymbol{E}}{\partial t} - \operatorname{rot} \boldsymbol{H} = -\operatorname{grad} e - \boldsymbol{j}_e
\end{aligned} \tag{2.9}$$

where $J_e^\mu = (\rho_e, \boldsymbol{j}_e)$ and $J_m^\mu = (\delta_m, \boldsymbol{j}_m)$ are external electric and magnetic currents, and $J = J_e + iJ_m$. The scalar field Z_0 satisfies the following equation:

$$\square Z_0 = \partial_\mu J^\mu \tag{2.10}$$

and then is not produced by conserved currents. Now one can introduce the vector potential $A = A^\mu \sigma_\mu$ by the following definition:

$$Z = \bar{\partial} A \tag{2.11}$$

and the Maxwell–Milner equation is equivalent to the following equation;

$$\partial \bar{\partial} A = J, \qquad \partial \bar{\partial} = \bar{\partial} \partial = \square. \tag{2.12}$$

We assume for simplicity that $A = A^\dagger$, i.e. there are no magnetic charges in the theory, but the formalism can also be developed without this restriction. For the complex potential $(A \neq A^\dagger)$ we obtain two kinds of photons, electric and magnetic photons, similar to particles and antiparticles in massive field theories.

Under the condition $A = A^\dagger$ we obtain the well known relations:

$$\begin{aligned}
&\boldsymbol{E} = -\operatorname{grad} A_0 - \partial_0 \boldsymbol{A} \\
&\boldsymbol{H} = \operatorname{rot} \boldsymbol{A} \\
&Z_0 = e = \partial_\mu A^\mu = \partial_0 A_0 + \operatorname{div} \boldsymbol{A}.
\end{aligned} \tag{2.13}$$

The gauge transformation has the form

$$A \mapsto A + \partial \Phi, \tag{2.14}$$

where $\Phi = \bar{\Phi} = \Phi^*$, a scalar field, and $\square \Phi = 0$.

3. The canonical formalism

The free field equation $\partial \bar{\partial} A = 0$ can be obtained using the following Lagrangian density:

$$\mathscr{L} = -\tfrac{1}{4}\langle (\bar{\partial} A)(\bar{A} \partial) + (\partial \bar{A})(A \bar{\partial}) \rangle \qquad A = A^\dagger. \tag{3.1}$$

One can check that

$$\begin{aligned}
&\mathscr{L} = \tfrac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{H}^2 - e^2) = \mathscr{L}_0 - \tfrac{1}{2}(\partial_\mu A^\mu)^2 \\
&\mathscr{L}_0 = \tfrac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{H}^2).
\end{aligned} \tag{3.2}$$

$\mathscr{L}$ is the well known Fermi Lagrangian which is non-singular in contrast to $\mathscr{L}_0$.

The field π_μ, canonically conjugate to the field A^μ, is defined as

$$\pi_0 = \partial\mathscr{L}/\partial\dot{A}_0 = -(\partial_0 A_0 + \operatorname{div} \boldsymbol{A}) = -e \tag{3.3}$$

$$\boldsymbol{\pi} = \partial\mathscr{L}/\partial\dot{\boldsymbol{A}} = \partial_0\boldsymbol{A} + \operatorname{grad} A_0 = -\boldsymbol{E}. \tag{3.4}$$

The Hamiltonian is equal to

$$H = \int \mathrm{d}^3\boldsymbol{x}\mathscr{H}(\boldsymbol{x}, t)$$

$$= \tfrac{1}{2}\int \mathrm{d}^3\boldsymbol{x}[(\partial_0\boldsymbol{A})^2 - (\partial_0 A_0)^2 + (\operatorname{div} \boldsymbol{A})^2 - (\operatorname{grad} A_0)^2 + (\operatorname{rot} \boldsymbol{A})^2] \tag{3.5}$$

where

$$\mathscr{H}(\boldsymbol{x}, t) = \sum_{\alpha=0}^{3} \pi_\alpha \dot{A}^\alpha - \mathscr{L}. \tag{3.6}$$

Now the canonical formalism can be developed in a standard way.

4. The quantum theory

The quantum theory of Maxwell–Milner electrodynamics in a vacuum will be built up by using the method of canonical quantisation. In the place of the canonical variables of classical theory, A^μ, π_μ, we shall introduce the field operators $\hat{A}^\mu$, $\hat{\pi}_\mu$ acting in the Hilbert space of the state vectors. Field operators taken at the same instant of time t satisfy the following commutation relations (CCR):

$$\begin{aligned} &[\hat{A}_\alpha(\boldsymbol{x}, t), \pi_\beta(\boldsymbol{y}, t)] = \delta_{\alpha\beta}\delta(\boldsymbol{x} - \boldsymbol{y}), \\ &[\hat{A}_\alpha(\boldsymbol{x}, t), \hat{A}_\beta(\boldsymbol{y}, t)] = [\hat{\pi}_\alpha(\boldsymbol{x}, t), \hat{\pi}_\beta(\boldsymbol{y}, t)] = 0. \end{aligned} \tag{4.1}$$

One can compute the CCR for the field operators in a covariant form

$$[\hat{A}_\mu(x), \hat{A}_\nu(x')] = \mathrm{i}g_{\mu\nu}D(x - x'). \tag{4.2}$$

Introducing the Weyl form of CCR (4.2) one can construct the c^*-algebra $\mathfrak{A}$ of observables for the vector potential (Carey, *et al* 1977). This is obviously the same algebra as in Maxwell theory and the difference lies in the physical interpretation. In Maxwell theory there are two methods of eliminating the unnecessary degrees of freedom. The first distinguishes the Hilbert space of physical states $\mathscr{H}_{\text{phys}}$ in the Hilbert space $\mathscr{H}$ related to the representation of $\mathfrak{A}$ by introducing the non-positive definite scalar product (Gupta-Bleuler formalism, see Strocchi and Wightman 1974). The physical space $\mathscr{H}_{\text{phys}}$ has the structure of the quotient space; $\mathscr{H}_{\text{phys}} = \mathscr{H}/\mathscr{H}_0$ where

$$\mathscr{H}_{\text{phys}} = \{\psi \in \mathscr{H}; \langle\psi|\partial_\mu\hat{A}^\mu(x)|\psi\rangle = 0, x \in M_4\}. \tag{4.3}$$

In the second method (Fermi quantisation) one can construct the representation of the physical algebra of observables $\mathfrak{A}_{\text{phys}}$ which has the structure of a factor algebra; $\mathfrak{A}^{\text{phys}} = \mathfrak{A}/I$ (see Carey *et al* 1977 for details). In Maxwell–Milner theory the full algebra $\mathfrak{A}$ has physical meaning. There exists an interesting subalgebra $\mathfrak{A}_0$ generated by fields $\hat{e}^{(\pm)}(x) = \partial_\mu\hat{A}^{\mu(\pm)}(x)$. Using (4.2) we obtain the relations

$$[\hat{e}^{(\pm)}(x), \hat{e}^{(\pm)}(x')] = 0 \tag{4.4}$$

and then $\mathfrak{A}_0$ is a complex Abelian subalgebra of $\mathfrak{A}$. It follows that the additional field $e(x)$ has a classical character. The problem of a concrete representation of $\mathfrak{A}$ in the Hilbert space has a non-unique solution and will not be discussed here.

5. The massive vector field

The massive vector field theory can also be modified in the same manner as electrodynamics by omitting the Lorentz condition $\partial_\mu A^\mu = 0$. We can use the Lagrangian density (3.1) with the additional mass term $\frac{1}{2}m^2\langle \bar{A}\, A\rangle$ and repeat the procedure of canonical quantisation. In contrast to the standard theory, all components of the vector field A^μ are independent canonical variables and the algebra of observables contain an Abelian subalgebra generated by the 'classical' field $\hat{e}(x) = \partial_\mu A^\mu(x)$. This modified theory can also be formulated for complex vector fields.

The above formulation of the vector field theory (massive or massless) leads to Hamiltonians which are unbounded from below but we are not sure that this property must be necessary forbidden.

6. The non-conservation of charge

The Milner theory of vector fields (massive or massless) admits an interaction of the vector field with an external non-conservative current. In that case the equation of motion has the form

$$(\Box - m^2)A^\mu = J^\mu \tag{6.1}$$

and then

$$(\Box - m^2)e = \partial_\mu J^\mu. \tag{6.2}$$

As a simple example we study the electromagnetic field ($m = 0$) generated by a particle with a charge q which was created at a point $\boldsymbol{x}_0 = 0$ and at time $t_0 = 0$.

Then we have

$$\partial_\mu J^\mu = q\delta(x) = q\delta(t)\delta(\boldsymbol{x}). \tag{6.3}$$

The current J^μ in the particle's rest frame has the following form:

$$J^\mu = (J^0, 0, 0, 0), \qquad J^0 = q\Theta(t)\delta(\boldsymbol{x}) \tag{6.4}$$

One can easily obtain from (6.2) that

$$e(x) = qD_{\mathrm{R}}(x) = \frac{q}{2\pi}\Theta(t)\delta(x_\mu x^\mu) = \frac{q}{4\pi}\frac{\delta(t - |\boldsymbol{x}|)}{|\boldsymbol{x}|} \tag{6.5}$$

and

$$A_0(\boldsymbol{x}, t) = \frac{q}{4\pi}\int_{-0}^{t}\frac{\delta(\tau - |\boldsymbol{x}|)}{|\boldsymbol{x}|}\,\mathrm{d}\tau \tag{6.6}$$

$$\boldsymbol{E} = -\operatorname{grad} A_0 = \frac{q}{4\pi}\begin{Bmatrix} \dfrac{\boldsymbol{x}}{|\boldsymbol{x}|^3}; & |\boldsymbol{x}| < t \\ 0; & |\boldsymbol{x}| \geqslant t \end{Bmatrix} + \frac{q}{4\pi}\frac{\delta(t - |\boldsymbol{x}|)}{|\boldsymbol{x}|^2}\boldsymbol{x}$$

$$\boldsymbol{E} = \boldsymbol{E}_{\text{Coulomb}} + \boldsymbol{E}_{\text{wave}}.$$

This electromagnetic field consists of the common Coulomb electric field and the spherical wave which contains the scalar field e, and the longitudinal electric field $\boldsymbol{E}_{\text{wave}}$. Now one can compute the energy $\mathscr{E}(t)$ of the field using the Hamiltonian H_1 of the electromagnetic field interacting with an external current:

$$H_1(t)=\int \mathrm{d}^3\boldsymbol{x}\mathscr{H}_1(\boldsymbol{x},t) \tag{6.7}$$

where

$$\mathscr{H}_1(\boldsymbol{x},t)=\mathscr{H}(\boldsymbol{x},t)-J^{\mu}(\boldsymbol{x},t)A_{\mu}(\boldsymbol{x},t) \tag{6.8}$$

and $\mathscr{H}(\boldsymbol{x},t)$ is given by (3.5). After the simple calculation we obtain (formally):

$$\mathscr{E}(t\to\infty)=\mathscr{E}_{\text{Coulomb}}+\mathscr{E}_{\text{wave}} \tag{6.9}$$

$$\mathscr{E}_{\text{Coulomb}}=\left(\frac{q}{4\pi}\right)^2\int \mathrm{d}^3\boldsymbol{x}\frac{1}{|\boldsymbol{x}|^4}=\frac{q^2}{4\pi}\int_0^{\infty}\frac{1}{r^2}\mathrm{d}r \tag{6.10}$$

$$\mathscr{E}_{\text{wave}}=-\frac{q^2}{4\pi}\int_{-\infty}^{+\infty}\delta^2(r)\,\mathrm{d}r. \tag{6.11}$$

The integrals (6.10), (6.11) are divergent because of the point-like structure of the particle and the rapid creation of it. However one can regularise them by introducing a smooth form factor $\delta^{\text{reg}}(x)$ in place of $\delta(x)$ in (6.3). The most important fact is that the energy of a spherical wave is negative. It follows that we can study the field theoretical models for which the charge $Q=\int d^3\boldsymbol{x}J^0$ can be produced from a 'vacuum' and the unboundedness of the Hamiltonian of the vector field allows creation of massive particles carrying that charge. The mass M of this particle is equal to:

$$M=-\mathscr{E}^{\text{reg}}_{\text{wave}} \qquad \text{and} \qquad M=\mathscr{E}^{\text{reg}}_{\text{Coulomb}}+\delta M$$

where δM is a non-electromagnetic mass.

7. Conclusion

The generalisation of electrodynamics (and massive vector field theory) presented above was suggested by the mathematical symmetry in a quaternionic notation and the possibility of canonical quantisation without any constraints. The physical significance of this formulation is an open question and will be studied in the future. The author hopes that the possibility of interaction with non-conserved currents should help in the explanation of the charge asymmetry of the visible Universe.

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